Convex Spherical Cubes

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The main point of this document is to also have some pictures, almost all were taken from this article by Guy Kindler, Anup Rao, Ryan O'Donnell, Avi Wigderson about their work on spherical foams.

1 Background on Kelvin's foam problem

The starting point for this work is the so-called *Kelvin's foam problem*, first studied by Lord Kelvin: how should we partition $\mathbb{R}^n$ into "bubbles" of volume 1 that minimizes average surface area?

Example 1.1.
- *In 2 dimensions we could take the square latice which corresponds to a "bubble" with perimeter 4. It was conjectured by the Greeks that the hexagonal honecomb lattice is optimal and this was proven by Hales* [4].
- *In 3 dimension we could take a cube which corresponds to a "bubble" with surface area 6.*

  Another way to think about how to cnstruct such bubbles is to build a lattice and then to take the Voronoi cells *around each lattice point.* (*Recall that the Voronoi cell of a lattice point is the bubble that includes all points closer to this point than to any other lattice point.*) *Lord Kelvin proposed that the optimal solution in 3 dimensions was a slightly distorted form of the Voronoi foam given by the body-centered cubic lattice. This truncated octahedron has a surface area of about 5.306.*

  Some physicists (*Weaire and Phelan*) *ultimately found an even more optimal foam based on the crystal structure of some real-world crystal. Of note is that their foam no longer consists of translations of the same shape. Their construction gives an average surface area of about 5.288. It is still an open question if this is the optimal construction. There are more descriptions about this foam in the linked article by Kindler, Rao, O'Donnell and Wigderson.*

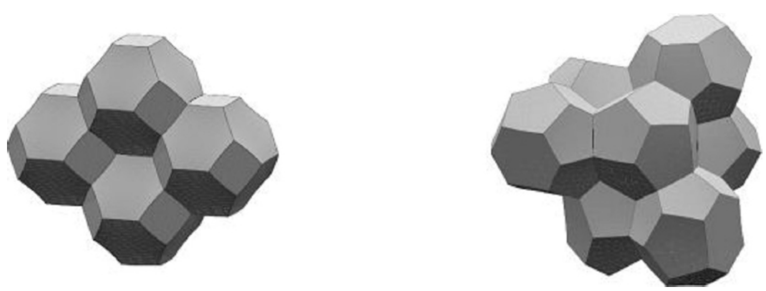


Figure 1: On the left is the Kelvin foam given by a distorted Voronoi foam of the body-centered cubic lattice. On the right is Weaire and Phelan's foam.

It's natural to ask this question asymptotically. Here are some relatively easy bounds we can obtain:

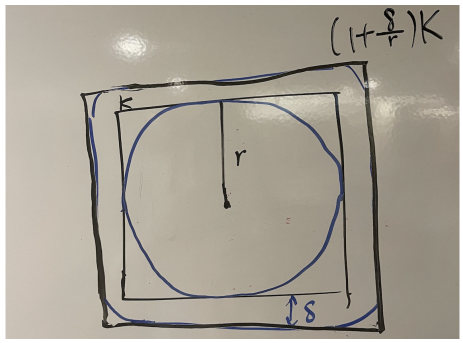
- (Lower bound) By the isoperimetric inequality, we know that any cell of volume 1 has surface area $\gtrsim \sqrt{n}$.
- (Upper bound) There's an easy upper bound by considering the unit cube, which gives a surface of $2n$.

Now, we can ask which of the above bounds is actually closer to the truth. It turns out that the $\sqrt{n}$ bound we get from isoperimetric inequality is actually tight. Let us begin by seeing a proof for this fact because it will introduce a key lemma that we will use to prove the main theorem.

Lemma 1.2 ([7, Lemma 3])**.** *Fix $n \in \mathbb{N}$ and $R > 0$. Suppose that a convex body $K \subset \mathbb{R}^n$ satisfies $K \supset RB^n$. Then,*

$$\frac{\mathrm{vol}_{n-1}(\partial K)}{\mathrm{vol}_n(K)} \leq \frac{n}{R}.$$

A picture speaks a thousand words so here is a (horribly drawn) pictorial depiction.



Proof. We can write:

$$\begin{aligned}
\mathrm{vol}(\partial K) &= \lim_{\delta \to 0} \frac{\mathrm{vol}((K + \delta B^n) \setminus K)}{\delta} \\
&\leq \lim_{\delta \to 0} \frac{\mathrm{vol}((1 + \delta/r)K \setminus K)}{\delta} \\
&= \frac{n}{r} \cdot \mathrm{vol}(K).
\end{aligned}$$

□

The slogan to remember is that containing a big ball implies a good bound on surface area. More on this in a bit, but at first glance $\frac{\mathrm{vol}_{n-1}(\partial K)}{\mathrm{vol}_n(K)}$ is an odd quantity to consider since it is not scale-invariant. Anyways, let us now see a proof for the tightness of the isoperimetric bound in this context of foams.

Theorem 1.3 (Folklore)**.** *The minimum average surface area of a foam in $\mathbb{R}^n$ is $\Theta(\sqrt{n})$.*

Proof. We can introduce a probability measure on the space of lattices as follows: identify the space $\mathcal{L}_n$ of all lattice of covolume 1 with $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$. Let μ_n be the unique Haar measure on $\mathcal{L}_n$ that is $SL_n(\mathbb{R})$-invariant. Then it is well-defined to consider a random covolume 1 lattice Λ according to μ_n. Last time I talked about the following theorem from the geometry of numbers:

Theorem 1.4 (Siegel's integral formula)**.** *For compactly supported, bounded and measurable $f : \mathbb{R}^n \to \mathbb{R}$, we have*

$$\int_{\mathcal{L}_n} \sum_{x \in L \setminus \{0\}} f(x) d\mu_n(L) = \int_{\mathbb{R}^n} f(x) dx.$$

Unrigorously, what Siegel's formula tells us is the distribution of lattice points in a random lattice is "as expected". In particular, if we take f to be the characteristic function of the ball of radius r centered at the origin, then the above basically says that on average a random lattice has $V(r)$(:= volume of a ball with radius r) nonzero vectors of length r.

Definition 1.5. For a lattice $\Lambda \subset \mathbb{R}^n$, write $\lambda_1(\Lambda)$ to be the length of the shortest vector in Λ: so $\lambda_1(\Lambda) = \min\{\|x\|_2 : x \in \Lambda \setminus \{0\}\}$.

Informally, the above tells us that with high probability $\lambda_1(\Lambda) \gtrsim \sqrt{n}$. It is not too difficult to make this precise.

The foam construction we will take is the *Voronoi cell* construction mentioned in the examples above. Formally, we have the following definition of the Voronoi cell as the set of poitns in space for which 0 is the closest lattice point.

Definition 1.6. For a lattice $\Lambda \subset \mathbb{R}^n$, define the Voronoi cell to be

$$\mathrm{Vor}(\Lambda) = \{y \in \mathbb{R}^n : \mathrm{dist}(y, \Lambda) = \|y\|\}.$$

In particular, note that $\mathrm{Vor}(\Lambda)$ contains a ball of radius $\lambda_1(\Lambda)$. By our earlier bound on $\lambda_1(\Lambda)$ and then plugging this into Lemma 1.2, it follows that the surface area of $\mathrm{Vor}(\Lambda)$ is $\lesssim \sqrt{n}$ as desired. □

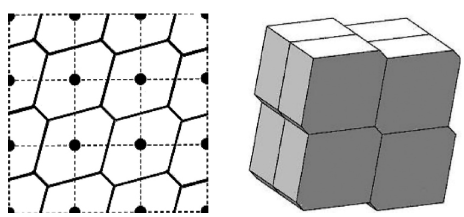
Moving forwards, our strategy would be to find lattices that do not contain short vectors and then to take suitable Voronoi cells of such a lattice.

2 Background on the cubical foam problem

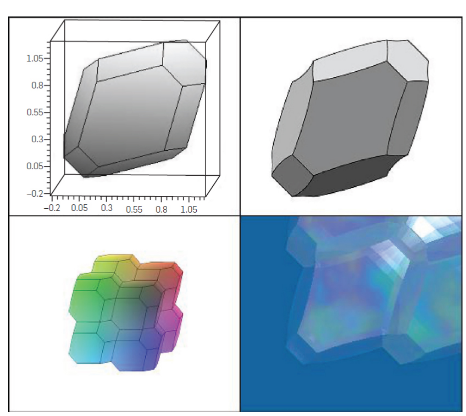
In this paper, the main problem studied is a restriction of the Kelvin foam problem. More precisely, they consider foams that are periodic with respect to the integer lattice.

> *Cubical foam problem*: What is the least surface area of a bubble that partitions $\mathbb{R}^d$ periodically according to the integer lattice?

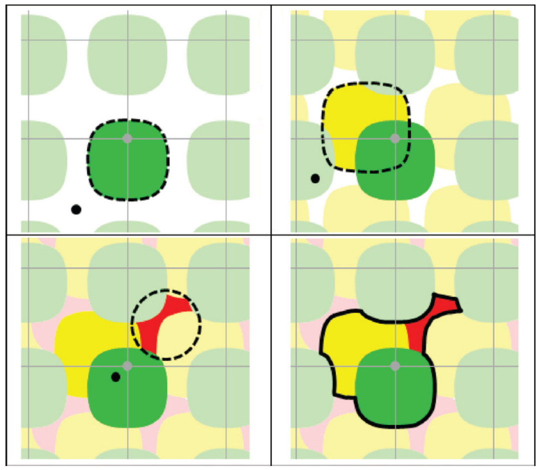
Example 2.1.
For the $n = 2$ case, the hexagonal lattice no longer works because we are restricting to consider tilings that are periodic with respect to the integer lattice. The square lattice still orks with perimeter 4. However, Choe [3] showed that the optimal solution is a slightly distorted form of the hexagonal lattice with perimeter 3.864.



Then optimal construction for the $n = 3$ case is still open. One possible construction is to "add thickness" to Choe's isosceles hexagon construction. However, Kindler, Rao, O'Donnell, Widgerson [6] gave the following better construction of a bubble with surface area about 5.602.



Once again, we can ask this same question asymptotically. It was shown in [6] and [1] that the answer, as in the case of the Kelvin foam problem, is $\Theta(\sqrt{n})$. We very briefly outline the method that [6] adopts. The vague idea is to take a body produced via a procedure called Holenstein's Consistent Sampling Lemma [5], translate it according to a random shift and then at each iteration we add to the body all the uncovered parts that the new translate to the original body. Here is a pictorial description of this process, as always taken from [8].



Remark 2.2. In relation to the parallel repetition problem, Braverman and Minzer [2] showed that if we imposed axis-symmetry conditions to the cubical foam problem we obtain a bound $\Theta(n/\sqrt{\log n})$.

3 On convex spherical cubes

The goal in this section is to prove the following.

Theorem 3.1 ([7, Theorem 1])**.** *For every $n \in \mathbb{N}$ there is a convex spherical cube with surface area $n^{1/2+o(1)}$.*

We'll study the quantity in Lemma 1.2 slightly more and then give a very high level overview for the strategy for the proof before delving into more details.

Even though the quantity $\mathrm{vol}_{n-1}(\partial K)/\mathrm{vol}_n(K)$ (for simplicity of notation I may sometimes drop the indices for the dimension) for a convex body K is not scale-invariant, the nice property that it has is that it is additive under product of convex bodies.

Lemma 3.2 ([7, Lemma 7])**.** *If K_1 and K_2 are convex bodies such that $K = K_1 \times K_2$ then*

$$\frac{\mathrm{vol}(\partial K)}{\mathrm{vol}(K)} = \frac{\mathrm{vol}(\partial K_1)}{\mathrm{vol}(K_1)} + \frac{\mathrm{vol}(\partial K_2)}{\mathrm{vol}(K_2)}.$$

Proof. This follows from rearranging $\mathrm{vol}(\partial K) = \mathrm{vol}(\partial K_1)\mathrm{vol}(\partial K_2) + \mathrm{vol}(K_1)\mathrm{vol}(\partial K_2)$. □

The high level strategy that [7] use is to in some sense take a subspace $V \subset \mathbb{R}^n$, project away from V to get a sub-lattice of $\mathbb{Z}^n$, sparsify this sub-lattice to get rid of short vectors and then we get a Voronoi cell with small boundary (in the same spirit as the proof of Theorem 1.3) and then recursively apply this same procedure to $\mathbb{Z}^n \cap V$. Ultimately we can take the bubble to be the product of the convex bodies given by the Voronoi cells of these sparse lattices and use the above additivity property of Lemma 3.2 to get a bound on the boundary of the bubble.

Now let us expand upon this proof strategy in slightly greater detail. A disclaimer is that in the following write-up, I'm going to be imprecise in many places and happily lose $n^{o(1)}$ factors so as to better convey the gist of the argument rather than being bogged down by technical details. The main lemma that we need for the purposes of sparsification is the following one on binary matrices with sparse rows and s-wise independent columns.

Lemma 3.3 ([7, Lemma 12])**.** *Let n be a positive integer. For $s \ll n$ and $m \leq n$ such that $n = \Omega(m \log m)$, there exists a matrix $A \in \mathrm{Mat}_{m\times n}(\{0,1\})$ with the following properties:*

- *Any s of the columns of A are linearly independent over $\mathbb{F}_2$*
- *Every row of A has at most $O(n/m)$ entries.*

Proof. To construct A, we sample each column as a random vector with $d = O(1)$ many entries that are 1. We can then calculate that with high probability the second bullet above is satisfied, and a first moment calculation also gives the first bullet above.

We'll follow the analysis in [7] because it is pretty neat. Write $m = s \log n$ and let $W(d)$ be a vector obtained by doing a random walk on the Boolean hypercube $\{0,1\}^m$. We'll consider the random matrix A whose columns are given by m copies of i.i.d. $W(d)$. Note that the probability that the ith row has an entry 1 in the jth position is given by $1-(1/m)^d \leq d/m$ because for it to be a 1 we need to have at least flipped the jth position once on the random walk. In other words, the probability that the ith row has at most $\ell \sim n/m$ entries is then at least $1 - m\binom{n}{\ell}(d/m)^\ell \geq 1 - m(edn/(m\ell))^\ell \gg 1$.

Next, we need to check that for any s of the columns $C_1, \ldots, C_n$ of A are linearly independent. Let κ be the number of $S \subset [n]$ with $|S| \leq s$ such that $\sum_{i\in S} C_i \neq \mathbf{0} \pmod 2$. We claim that it suffices to show that $\mathbb{E}[\kappa] < 1/10$. This is because by Markov's inequality it would then follow that $\mathbb{P}[\kappa = 0] = 1 - \mathbb{P}[\kappa = 1] > 9/10$.

Before we proceed, note that there is a nice property of this current set-up which is that if we have t i.i.d. copies $W_1(d), \ldots, W_t(d)$ of $W(d)$ then $\sum_{i=1}^t W_i(d)$ has the same distribution as $W(td)$. Using this property, it follows that

$$\mathbb{E}[\kappa] = \sum_{r=1}^{s} \binom{n}{r} \mathbb{P}[W(dr) = \mathbf{0}].$$

To finish up, we use the bound.

Claim 3.4. $\mathbb{P}[W(t) = \mathbf{0}] \lesssim (t/m)^{t/2}$

Proof. We convert this to something we can apply Chernoff-Hoeffding on:

$$\begin{aligned}
\mathbb{P}[W(t) = \mathbf{0}] &= \frac{1}{2^m} \sum_{k=0}^{m} \binom{m}{k} \left(1 - \frac{2k}{m}\right)^t \\
&= \left(-\frac{2}{m}\right)^t \mathbb{E}_{x_i \text{ Bernoulli}} \left[\left(\sum_{i=1}^{m} x_i - \frac{m}{2} \right)^t \right] \\
&= \left(\frac{2}{m}\right)^t \int_0^\infty t y^{t-1} \mathbb{P}\left[\sum_{i=1}^{m} x_i \geq y + \frac{m}{2} \right] dy \\
&\leq 2t \left(\frac{2}{m}\right)^t \int_0^\infty y^{t-1} e^{-2y^2/m} dy \\
&\sim \left(\frac{2}{m}\right)^t (t/2)! \sim (t/m)^{t/2}.
\end{aligned}$$

□

Plugging in this bound, it follows that

$$\begin{aligned}
\mathbb{E}[\kappa] &\leq 2 \sum_{r=1}^{s} \left(\frac{e d^{d/2} r^{d/2-1} n}{m^{d/2}} \right)^r \\
&\leq 2 \sum_{r=1}^{s} \left(\frac{e d^{d/2} r^{d/2-1} n}{m^{d/2}} \right)^r \\
&\leq 2 \sum_{r=1}^{s} (0.01)^r < 1/10.
\end{aligned}$$

□

Remark 3.5. In [7], it is mentioned that this kind of argument is common in works about Low Density Parity Check (LDPC) codes. I'm not too familiar with this line of work but if anybody has any ideas about these literature please let me know. I'm specifically interested in if there are derandomized ways to construct these sorts of "sparse" matrices, because this is the only source of randomness in the entire proof.

Proof of Theorem 3.1. We induct on n. For small values of the dimension n, it suffices to use the square grid bound of $\sim n$.

Let A be the matrix given in Lemma 3.3 with $m = n^{1-o(1)}$ and $s = n^{1-o(1)}$ where we pick parameters suitably. Apriori A may not have rank m. But we can make some cosmetic changes like pick the r rows that are linearly independent and then append $m - r$ rows of the basis vectors to A and it's not difficult to see that we still maintain the properties of Lemma 3.3. We'll work with this modified matrix and in an abuse of notation still refer to it as A.

Consider the subspace V given by the rowspan of A. The key point here is that the linear independent condition on the rows of A translates into:

any element of $\mathbb{Z}^n \cap V^\perp$ has at least s nonzero entries.

In particular this means that we can bound the length of the shortest vector in the sub-lattice as $\lambda_1(\mathbb{Z}^n \cap V^\perp) > \sqrt{s}$. Using Lemma 1.2 on $K_1 = \mathrm{Vor}(\mathbb{Z}^n \cap V^\perp)$, then

$$\frac{\mathrm{vol}(\partial K_1)}{\mathrm{vol}(K_1)} \lesssim n/\sqrt{s} = n^{1/2+o(1)}.$$

We need to somehow lift this construction to the entire space. So we need to handle the part of $\mathbb{Z}^n$ projected onto V, with we denote as $\mathrm{proj}_V(\mathbb{Z}^n)$. Now, since the rows of A have length $\sim \sqrt{n/m}$, it follows that $\mathrm{proj}_V(\mathbb{Z}^n) \sim \sqrt{m/n}\mathbb{Z}^m$. On $\mathbb{Z}^m$ we can get a convex body with corresponding Q_2. Scaling this convex body appropriately in $\mathrm{proj}_V(\mathbb{Z}^n)$, we can get a convex body K_2 such that

$$\frac{\mathrm{vol}(\partial K_2)}{\mathrm{vol}(K_2)} \sim \sqrt{n/m}Q(m) \sim n^{1/2+o(1)}.$$

Consequently, if we take $K = K_1 \times K_2$ to tile $\mathbb{Z}^n$, we can apply Lemma 3.2 to combine the above estimates and get

$$\frac{\mathrm{vol}(\partial K)}{\mathrm{vol}(K)} = \frac{\mathrm{vol}(\partial K_1)}{\mathrm{vol}(K_1)} + \frac{\mathrm{vol}(\partial K_2)}{\mathrm{vol}(K_2)} \lesssim n^{1/2+o(1)}.$$

□

References

[1] Noga Alon and Bo'az Klartag, *Economical toric spines via Cheeger's inequality*, J. Topol. Anal. **1** (2009), no. 2, 101–111. MR 2541756

[2] Mark Braverman and Dor Minzer, *Optimal tiling of the Euclidean space using permutation-symmetric bodies*, 36th Computational Complexity Conference, LIPIcs. Leibniz Int. Proc. Inform., vol. 200, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2021, pp. Art. No. 5, 48. MR 4287078

[3] Jaigyoung Choe, *On the existence and regularity of fundamental domains with least boundary area*, J. Differential Geom. **29** (1989), no. 3, 623–663. MR 992333

[4] T. C. Hales, *The honeycomb conjecture*, Discrete Comput. Geom. **25** (2001), no. 1, 1–22. MR 1797293

[5] Thomas Holenstein, *Parallel repetition: simplifications and the no-signaling case*, Theory Comput. **5** (2009), 141–172. MR 2559656

[6] Guy Kindler, Ryan O'Donnell, Anup Rao, and Avi Wigderson, *Spherical cubes and rounding in high dimensions*, 2008 49th annual ieee symposium on foundations of computer science, IEEE, 2008, pp. 189–198.

[7] Assaf Naor and Oded Regev, *An integer parallelotope with small surface area*, arXiv preprint arXiv:2301.02862 (2023).

[8] Or Ordentlich, Oded Regev, and Barak Weiss, *New bounds on the density of lattice coverings*, J. Amer. Math. Soc. **35** (2022), no. 1, 295–308. MR 4322394