Stochastic Calculus and Applications – Based off Jason Miller's Part III class

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In the first part, we develop a proper theory of Itô's integration i.e. how do we make sense of

$$(H \cdot X)_t \text{``}=\text{''} \lim_{\varepsilon \to 0} H_{k\varepsilon}(X_{(k+1)\varepsilon} - X_{k\varepsilon})$$

by exploiting the orthogonality of martingale increments in X to get cancellation, even if X itself has rather rough sample paths.

1 "Normal" integrals

Definition 1.1 (Finite variation)**.** Let $a\colon [0,\infty) \to \mathbb{R}$ be a cadlag function. For any $n \in \mathbb{N}$ and $t \geq 0$, let

$$v^n(t) = \sum_{k=0}^{\lceil 2^n t \rceil - 1} |a((k+1)2^{-n}) - a(k2^{-n})|.$$

Then $v(t) := \lim_{n\to\infty} v^n(t)$ exists for all t and is the total variation of a on $(0,t]$.

Claim 1.2. *A cadlag function $a\colon [0,\infty) \to \mathbb{R}$ can be written as a difference of two right-continuous, non-decreasing functions if and only if a is of finite variation.*

Proof. Let $a^+ = \frac{1}{2}(v+a)$ and $a^- = \frac{1}{2}(v-a)$ and then note that

$$a^+(t) - a^+(s) = \lim_{n\to\infty} \frac{1}{2}\left[\sum_{k=2^n s_n^+}^{2^n t_n^- - 1} (|a((k+1)2^{-n}) - a(k2^{-n})| + (a((k+1)2^{-n}) - a(k2^{-n}))) \right.$$
$$\left. + |a(t_n^+) - a(t_n^-)| + (a(t_n^+) - a(t_n^-))\right].$$

□

Claim 1.3. *Let A be a cadlag, adapted process of finite variation V. Then V is cadlag, adapted and pathwise non-decreasing.*

To see V is adapted, note that V_t is the limit of $\mathcal{F}_t$-measurable RVs:

$$V_t = \lim_{n\to\infty} \sum_{k=0}^{2^n t_n^- - 1} |A_{(k+1)2^{-n}} - A_{k2^{-n}}| + |\Delta A_t|.$$

Definition 1.4. The previsibile σ-algebra $\mathcal{P}$ on $\Omega \times (0,\infty)$ is the σ-algebra generated by sets of the form $E \times (s,t]$ for $E \in \mathcal{F}_s$ and $s < t$. Then we say $H\colon \Omega \times (0,\infty)$ is *previsible* if it is measurable w.r.t. $\mathcal{P}$.

Claim 1.5. *Let X be a cadlag adapted process, and define for all t the process $H_t := X_{t^-}$. Then H is previsible.*

Whsn is a process previsible?

Claim 1.6. *A process H is previsible, then H_t is previsible with respect to $\mathcal{F}_{t^-}$*

First use monotone class theorem to establish that the vector space

$$V = \{H\colon \Omega \times [0,\infty) \to \mathbb{R} \text{ s.t. } H_t \text{ is } \mathcal{F}_t\text{-measurable } \forall t \geq 0\}$$

contains all measurable bounded functions. And then use standard approximation techniques: consider $H_n = \frac{\lfloor 2^n H \rfloor}{2^n} \wedge n$.

Example 1.7. A Poisson process $(N_t)_{t\geq 0}$ is not previsible since N_t is not $\mathcal{F}_{t^-}$-measurable.

Theorem 1.8. *Let $A\colon \Omega \times [0,\infty) \to \mathbb{R}$ be a cadlag, adapted process of finite variation V. Let H be a previsible process, and assume that for all $\omega \in \Omega$ it holds that for all $t > 0$:*

$$\int_{(0,t]} |H(\omega, s)| dV(\omega, s) < \infty.$$

Then $(H \cdot A)(\omega, t) := \int_{(0,t]} H(\omega, s) dA(\omega, s)$ is cadlag, adapted and of finite variation.

Proof.

- Well-definedness follows by writing $H \cdot A = H^+ \cdot A^+ - H^- \cdot A^+ - H^+ \cdot A^- + H^- \cdot A^-$ and then invoking the condition to conclude that each of these terms are finite.
- It's not difficult to prove cadlag, also note that

$$\Delta(H \cdot A)_t = \int H_s \mathbf{1}(s = t) dA_s = H_t \Delta A_t.$$

- We prove adaptedness by a monotone class argument
- Finite variation follows by the earlier characterization of writing it as a difference of non-decreasing processes

□

2 Local Martingales

Definition 2.1. A filtration $(\mathcal{F}_t)_{t\geq 0}$ is said to satisfy *usual conditions* if:

- $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets,
- $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous, that is for all $t \geq 0$ it holds that $\mathcal{F}_t = \mathcal{F}_{t^+} := \bigcap_{s>t} \mathcal{F}_s$.

Remark 2.2. For continuous X, we define $\mathcal{F}_T := \{E \in \mathcal{F} : E \cap \{T \leq t\} \in \mathcal{F}_t,\ \forall t \geq 0\}$.

We can remember optional stopping theorem as effectively stating that *the class of cadlag martingales is stable under stopping.*

Definition 2.3 (Local martingales)**.** A cadlag adapted process is called a *local martingale* if there exists a sequence $(T_n)_{n\geq 1}$ of stopping times with $T_n \uparrow +\infty$ almost surely, such that the stopped process X^{T_n} is a martingale for all $n \geq 1$. In this case we say that $(T_n)_{n\geq 1}$ *reduces* X.

Remark 2.4. Consider $M_t = |B_t|^{-1}$. We can easily check that $\mathbb{E}[M_t] \to 0$ as $t \to \infty$, so it is not a martingale. But we claim that $T_n = \inf\{t \geq 1 : |B_t| \leq n^{-1}\}$ is a sequence of reducing times for M_t.

To show that $M_{t\wedge T_n}$ is a martingale, use the fact that $|x|^{-1}$ is harmonic and $f(B_t) - f(B_0) - \frac{1}{2}\int_0^t \Delta f(B_s)ds$ is a martingale. There is a need to carve out the singularity at 0.

To prove that $T_n \uparrow \infty$ it suffices to use *annulus probabilities*. That is, let $S_r = \inf\{t \geq 1 : |B_t| > R\}$ and then use $\mathbb{E}[M_{T_n \wedge S_R}] = \mathbb{E}[M_1]$ and write

$$\mathbb{P}(\lim_{n\to\infty} T_n < \infty) \leq \mathbb{P}(\exists R : T_n < S_R \text{ for all } n) = \lim_{R\to\infty} \lim_{n\to\infty} \mathbb{P}[T_n < S_R].$$

Claim 2.5. *If X is a local martingale and $X_t \geq 0$ for all $t \geq 0$ then X is a supermartingale.*

Proof.

$$\mathbb{E}[X_t \,|\, \mathcal{F}_s] = \mathbb{E}[\liminf_{n\to\infty} X_{t\wedge T_n} \,|\, \mathcal{F}_s] \leq \liminf_{n\to\infty} \mathbb{E}[X_{t\wedge T_n} \,|\, \mathcal{F}_s] = \liminf_{n\to\infty} X_{s\wedge T_n} = X_s.$$

□

When is a local martingale an actual martingale?

Claim 2.6. *TFAE:*

- *X is a martingale.*
- *X is a local martingale and for all $t \geq 0$, the family*

$$\mathcal{X}_t = \{X_T : T \text{ is a stopping time with } T \leq t\}$$

is UI.

Proof. ($\Rightarrow$) follows from OST which gives $X_T = \mathbb{E}[X_t \,|\, \mathcal{F}_T]$ and then using the fact that if $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ then $\{E[X \,|\, \mathcal{G}] : \mathcal{G} \subset \mathcal{F} \text{ is a sub-}\sigma\text{-algebra}\}$. Consequently, $\mathcal{X}_t$ is UI.

($\Leftarrow$) It suffices to show that for all bounded stopping times T we have $\mathbb{E}[X_T] = \mathbb{E}[X_0]$. To show this we use the UI martingale convergence theorem on

$$\mathbb{E}[X_0] = \mathbb{E}[X_0^{T_n}] = \mathbb{E}[X_T^{T_n}] = \mathbb{E}[X_{T\wedge T_n}].$$

□

Corollary 2.7. *A bounded local martingale is a (true) martingale.*

Theorem 2.8. *If X is a continuous local martingale with $X_0 = 0$ and X has finite variation then $X \equiv 0$ almost surely.*

Proof. Define $T_n = \inf\{t \geq 0 : V_t = n\}$. Then the sequence $(T_n)_{n\geq 1}$ reduces X, because $|X_t^{T_n}| \leq |V_{t\wedge T_n}| \leq n$ and apply Corollary 2.7). Let $Y = X^{T_n}$. We compute

$$\begin{aligned}
\mathbb{E}[Y_t^2] &= \mathbb{E}\left[\sum_{k=0}^{N-1} (Y_{t_{k+1}} - Y_{t_k})^2\right] \\
&\leq \mathbb{E}[\sup_{0\leq\leq N} |Y_{t_{k+1}} - Y_{t_k}| \cdot \sum_{k=0}^{N-1} |Y_{t_{k+1}} - Y_{t_k}|] \to 0,
\end{aligned}$$

where in the last step we used DCT and

$$\lim_{N\to\infty} \sup_{0\leq k\leq N} |Y_{t_{k+1}} - Y_{t_k}| = 0.$$

□

Remark 2.9. *Continuity* here is important! Counterexample otherwise: Let $N \sim \text{Poi}(1)$, and let $X_t = N_t - t$ for $t \geq 0$, then X is of finite variation and X is a martingale.

The following depends on the notion of quadratic variation which is introduced in a later section.

Claim 2.10 (Example sheet 2). *If M is a continuous local martingale with $M_0 = 0$, then M is a L^2-bounded martingale iff $\mathbb{E}[[M]_\infty] < \infty$.*

Proof. ($\Leftarrow$) follows by MCT + Doob's. Let T_n be a reducing sequence. Then

$$\begin{aligned}\mathbb{E}[\sup_t M_t^2] &= \lim_n \mathbb{E}\left[\sup_{t \leq T_n} M_t^2\right] = \lim_n \mathbb{E}\left[\sup_t (M_t^{T_n})^2\right] \\ &\leq 4 \lim_n \mathbb{E}[M_{T_n}^2] = 4 \lim_n \mathbb{E}[[M]_{T_n}] = 4\mathbb{E}[[M]_\infty] < \infty.\end{aligned}$$

□

Remark 2.11 (Some other example sheet style problems). The same proof shows that if for all times t we have $\mathbb{E}[[M]_t] < \infty$ then it follows that M_t is a martingale. Consider an Itô process Y (to be defined...) with Doobs-Meyer decomposition given by $dY_t = \beta(t)dt + \sigma(t)dX_t$ where X is a martingale. Then $[Y] = \sigma^2(t) \cdot [X]$ a.s. and so this shows for example that $\mathbb{E}\left[\int_0^t e^{\alpha B_s} dB_s\right] = 0$ since $\int_0^t e^{\alpha B_s} dB_s$ is a martingale.

Is there a "canonical" choice of reducing sequences?

Claim 2.12. *Let X be a continuous local martingale with $X_0 = 0$. For $n \geq 1$, define the stopping times*

$$T_n = \inf\{t \geq 0 : |X_t| = n\}.$$

Proof.

- (T_n are stopping times)

$$\{T_n \leq t\} = \left\{\sup_{s \in [0,t]} |X_s| \geq n\right\} = \bigcap_{k \geq 1} \bigcup_{\substack{s \leq t \\ s \in \mathbb{Q}}} \left\{|X_s| > n - \frac{1}{K}\right\} \in \mathcal{F}_t$$

- ($T_n \uparrow \infty$) Continuity of X implies $\sup_{s \in [0,t]} |X_s(\omega)| < \infty$, so there exists finite $n(\omega, t)$ such that $n(\omega, t) > \sup_{s \in [0,t]} |X_s(\omega)|$ which implies that $T_n(\omega) > t$.
- (T_n reduces X) Let T_n^* be a reducing sequence, then $X^{T_m \wedge T_n^*}$ is a martingale so that X^{T_n} is a local martingale, but since it is bounded it is a true martingale.

□

3 Itô integrals

Let $\mathcal{S}$ denote the set of simple processes

$$H(\omega, t) = \sum_{k=0}^{n-1} Z_k(\omega) \mathbf{1}_{(t_k, t_{k+1}]}(t)$$

for some $n \in \mathbb{N}$, and $0 = t_0 < t_1 < \cdots < t_n < \infty$ and Z_k is a bounded $\mathcal{F}_{t_k}$-measurable random variable. Also define:

- $\mathcal{M}^2 = \{L^2$ bounded, cadlag martingales$\}$
- $\mathcal{M}_c^2 = \{L^2$ bounded, continuous martingales$\}$
- $\mathcal{M}_{c,loc}^2 = \{L^2$ bounded, continuous local martingales$\}$

3.1 $H \in \mathcal{S}$, $M \in \mathcal{M}^2$

Claim 3.1. *Define for $H \in \mathcal{S}$ and $M \in \mathcal{M}^2$,*

$$(H \cdot M)_t := \sum_{k=0}^{n-1} Z_k (M_{t_{k+1}\wedge t} - M_{t_k \wedge t})$$

and then $H \cdot M \in \mathcal{M}^2$.

Proof. The L^2 boundedness follows from independence of increments and also Doob's maximal inequality. □

Claim 3.2. *Let $H \in \mathcal{S}$ and $M \in \mathcal{M}^2$. Then for any stopping time T, it holds that*

$$H \cdot M^T = (H \cdot M)^T.$$

3.2 $H \in L^2(M), M \in \mathcal{M}_c^2$

Now, we build towards Itô isometry to extend the definition of the stochastic integral from $\mathcal{S}$. But first we need to introduce Hilbert space structures to the integrators and integrand.

To equip the integrators $\mathcal{M}_c^2$ with a Hilbert space structure, for X a cadlag adapted process, define the norm

$$|\!|\!| X |\!|\!| = \|\sup_{t\geq 0} |X_t|\|_{L^2}$$

and let $\mathcal{C}^2 = \{\text{cadlag adapted processes } X \text{ with } |\!|\!| X |\!|\!| < \infty\}$. On $\mathcal{M}^2$, we also define $\|X\| := \|X_\infty\|_{L^2}$. The point is that $\mathcal{M}_c^2 = \mathcal{M}_c \cap \mathcal{M}^2$ is a *closed subspace*.

Another way to think about Doob's maximal inequality is that $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ are equivalent norms. Here's an example calculation to show that if (X^n) is a sequence in $\mathcal{M}^2$ such that $|\!|\!| X^n - X |\!|\!| \to 0$ then X is a martingale:

$$\begin{aligned}
\|\mathbb{E}[X_t \,|\, \mathcal{F}_s] - X_s\|_{L^2} &\leq \|\mathbb{E}[X_t - X_t^n \,|\, \mathcal{F}_s] + X_s^n - X_s\|_{L^2} \\
&\leq \|\mathbb{E}[X_t - X_t^n \,|\, \mathcal{F}_s]\|_{L^2} + \|X_s^n - X_s\|_{L^2} \\
&\leq \|X_t - X_t^n\|_{L^2} + \|X_s^n - X_s\|_{L^2} \\
&\leq 2 |\!|\!| X^n - X |\!|\!| \to 0.
\end{aligned}$$

Next we equip the integrands with a Hilbert space structure.

Definition 3.3 (UCP convergence)**.** Let (X^n) be a sequence of processes, then we say that $X^n \to X$ uniformly on compact in probability if for every $\varepsilon, t > 0$ we have

$$\mathbb{P}[\sup_{s\leq t} |X_s^n - X_s| > \varepsilon] \to 0.$$

Theorem 3.4 (Quadratic variation)**.** *Let M be a continuous local martingale. Then there exists a unique (up to indistinguishability) continuous, adapted and non-decreasing process $[M]$ such that $[M]_0 = 0$ and $M^2 - [M]$ is a continuous, local martingale. Moreover, if we define*

$$[M]_t^n = \sum_{k=0}^{\lceil 2^n t\rceil - 1} (M_{(k+1)2^{-n}} - M_{k2^{-n}})^2$$

then $[M]^n \to [M]$ ucp as $n \to \infty$.

Remark 3.5. The intuition here is that if X has *finite variation* then if we chop up the interval $[0,T]$ into $T/\delta t$ intervals of size δt then on each such smaller interval $(k\delta t,(k+1)\delta t)$ we increase X on the order $\sim \delta t$, and then adding $T/\delta t$ of some $\sim \delta t$ gives $O(1)$.

For quadratic variation, the intuition is to think about Brownian motion, where $\mathbb{E}|B_{t+\delta t} - B_t| = \sqrt{\delta t}|N(0,1)|$, so in particular if X has finite quadratic variation we should think that on the smaller interval $(k\delta t,(k+1)\delta t)$, X increases on the order $\sim \sqrt{\delta t}$ so that if we add $T/\delta t$ of *square of increment* then we get $O(1)$.

Proof.

- (Uniqueness) This follows from the fact that the only continuous local martingales wiht bounded variation is a.s. 0.

- (Existence $M \in \mathcal{M}_c^2$) The strategy is to guess what $M^2 - [M]$ should look like. Start with compact time sets $[0,T]$, where T is deterministic and finite. We can try to approximate M^T dyadically by

$$H_t^n = \sum_{k=0}^{\lceil 2^n T\rceil - 1} M_{k2^{-n}} \mathbf{1}_{(k2^{-n},(k+1)2^{-n}]}(t)$$

then dyadically we would build the martingale to look like

$$X_t^n = (H^n \cdot M)_t = \sum_{k=0}^{\lceil 2^n T\rceil - 1} M_{k2^{-n}}(M_{(k+1)2^{-n}\wedge t} - M_{k2^{-n}\wedge t}).$$

Indeed, we can check that $M_{k2^{-n}}^2 - 2X_{k2^{-n}}^n = \sum_{j=0}^{k-1}(M_{(j+1)2^{-n}} - M_{j2^{-n}})^2 = [M]_{k2^{-n}}^n$.

The first step is to show that (X^n) is bounded in $\|\cdot\|$. The only tool we have at our disposal is Cauchy-Schwarz to "isolate processes", so we just follow our nose, writing $H = H^n - H^m$:

$$\begin{aligned}
\|X^n - X^m\|^2 &= \mathbb{E}[(H\cdot M)_T^2] \\
&= \mathbb{E}\left[\sum_{k=0}^{\lceil 2^n T\rceil - 1} H_{k2^{-n}}^2 (M_{(k+1)2^{-n}} - M_{k2^{-n}})^2\right] \\
&\le \mathbb{E}\left[\sup_{t\in[0,T]} |H_t|^2 \sum_{k=0}^{\lceil 2^n T\rceil - 1} (M_{(k+1)2^{-n}} - M_{k2^{-n}})^2\right] \\
&\underset{\text{C.S.}}{\le} \mathbb{E}\left[\left(\sup_{t\in[0,T]} |H_t|^2\right)^2\right]^{1/2} \mathbb{E}\left[\left(\sum_{k=0}^{\lceil 2^n T\rceil - 1} (M_{(k+1)2^{-n}} - M_{k2^{-n}})^2\right)^2\right]^2 \\
&\lesssim \|M\|_{L^\infty}^C
\end{aligned}$$

for some constant $C > 0$.

Finally we should check that $[M]^n \to [M]$ ucp. So we first now that

$$\sup_{0\le t\le T} |X_t^n - Y_t| \to 0$$

in probability. It suffices to combine this with uniform continuity of M^2 and Y on $[0, T+1]$:

$$\sup_{0\le t\le T} |[M]_t - [M]_t^n| \le \sup_{0\le t\le T} |M_{2^{-n}\lceil 2^n t\rceil}^2 - M_t^2| + 2\sup_{0\le t\le T} |X_{2^{-n}\lceil 2^n t\rceil}^n - Y_{2^{-n}\lceil 2^n t\rceil}| + 2\sup_{0\le t\le T} |Y_{2^{-n}\lceil 2^n t\rceil} - Y_t|.$$

- (Existence $M \in \mathcal{M}_{c,loc}$) This is a localization argument. Define $T_n = \inf\{t \geq 0 : |M_t| \geq n\}$. Then apply the previous step to get a unique continuous adapted and non-decreasing process $[M^{T_n}]$ on $[0, \infty)$ so that $[M^{T_n}]_0 = 0$ and $(M^{T_n})^2 - [M^{T_n}] \in \mathcal{M}_{c,loc}$. Uniqueness allows us to "stitch" everything together.

  It remains to be seen that $[M]^n \to [M]$ ucp as $n \to \infty$, which follows by considering that when $\{T_k > T\}$ then we can just use the previous step, and $T_k \uparrow \infty$:

$$\mathbb{P}\left[\sup_{t\in[0,T]} |[M]_t^n - [M]_t| > \varepsilon\right] \leq \mathbb{P}[T_k \leq T] + \mathbb{P}\left[\sup_{t\in[0,T]} |[M^{T_k}]_t^n - [M^{T_k}]_t| > \varepsilon\right] \to 0.$$

□

Example 3.6. For the standard Brownian motion, we have $[B_t] = t$.

Corollary 3.7. *If $M \in \mathcal{M}_c^2$ then $M^2 - [M]$ is a UI martingale.*

Proof. We just keep using the fact that if we can bound a local martingale above by something integrable then we can conclude that we have a true martingale.

Let $S_n = \inf\{t \geq 0 : [M]_t \geq n\}$ for all $n \geq 1$. Then $S_n \uparrow +\infty$ and S_n is a stopping time for all n, with $[M]_{t\wedge S_n} \leq n$.

By Doob's we have that $M_{t\wedge S_n}^2 - [M]_{t\wedge S_n}$ is dominated by something integrable. Then combine OST and MCT (LHS)/DCT (RHS) to get:

$$\mathbb{E}[[M]_{t\wedge S_n}] = \mathbb{E}[M_{t\wedge S_n}^2] \implies \mathbb{E}[[M]_{S_n}] = \mathbb{E}[M_{S_n}^2].$$

Taking $n \to \infty$ and then we get that $[M]_\infty$ is integrable, and this is enough to show that $|M_t^2 - [M]_t|$ is dominated by an integrable random variable. □

Now that we have the non-decreasing function $[M](\omega)$, we can define a corresponding Lebesgue-Stieltjes measure; indeed, we can define μ on $\mathcal{P}$ the previsible σ-algebra by:

$$\mu(E \times (s,t]) = \mathbb{E}[\mathbf{1}(E)([M]_t - [M]_s)].$$

(and noting that $E \times (s,t]$ is a π-system that generates $\mathcal{P}$)

Then we define $L_2(M) := L^2(\Omega \times (0,\infty), \mathcal{P}, \mu)$ and write

$$\|H\|_{L^2(M)} = \|H\|_M := \left[\mathbb{E}\left(\int_0^\infty H_s^2 d[M]_s\right)\right]^{1/2}.$$

So far we have all the definitions in place to state Itô's isometry which allows us to extend our work in the earlier section. To spell it out, we first discuss the stochastic integral that we built by hand for $H \in \mathcal{S} \subset L^2(M)$. We claim that for any $M \in \mathcal{M}_c^2$, the map $H \mapsto H \cdot M$ provides an isometry between $(L^2(M), \|\cdot\|_M)$ and $(\mathcal{M}_c^2, \|\cdot\|)$ when restricted to $\mathcal{S} \subset L^2(M)$. Indeed, using the fact that $M^2 - [M]$ is a martingale

$$
\begin{aligned}
\|H \cdot M\|^2 &= \|(H \cdot M)_\infty\|_{L^2}^2 \\
&= \sum_{k=0}^{n-1} \mathbb{E}[Z_k^2 (M_{t_{k+1}} - M_{t_k})^2] \\
&= \sum_{k=0}^{n-1} \mathbb{E}[Z_k^2 \mathbb{E}[M_{t_{k+1}}^2 - M_{t_k}^2 \mid \mathcal{F}_{t_k}]] \\
&= \sum_{k=0}^{n-1} \mathbb{E}[Z_k^2 \mathbb{E}[[M]_{t_{k+1}} - [M]_{t_k} \mid \mathcal{F}_{t_k}]] \\
&= \sum_{k=0}^{n-1} \mathbb{E}[Z_k^2 ([M]_{t_{k+1}} - [M]_{t_k})] \\
&= \mathbb{E}\left[\int_0^\infty H_s^2 d[M]_s\right] = \|H\|_M^2.
\end{aligned}
$$

We can extend this to the entire of $L^2(M)$.

Theorem 3.8 (Itô's isometry)**.** *There exists a unique isometry $I \colon L^2(M) \to \mathcal{M}_c^2$ such that $I(H) = H \cdot M$ for all $H \in \mathcal{S}$.*

This is to be expected since $\overline{\mathcal{S}}$ contains indicator functions of all previsible processes $\mathcal{P}$, and so $\mathcal{S}$ is dense in $L^2(\mathcal{P}, \mu)$ for any choice of finite measure μ on $\mathcal{P}$.

Proof. Extending I to the whole of $L^2(M)$ follows because $I(\cdot)$ is linear for simple processes. To show there is isometry, use $\|\|x\| - \|y\|\| \leq \|x - y\|$ and get

$$
\|H \cdot M\| = \lim_{n\to\infty} \|H^n \cdot M\| = \lim_{n\to\infty} \|H^n\|_M = \|H\|_M.
$$

□

3.3 H locally bounded, $M \in \mathcal{M}_{c,loc}$

Claim 3.9. *Let $M \in \mathcal{M}_c^2$ and $H \in L^2(M)$ and let T be a stopping time, then*

$$
(H \cdot M)^T = (H\mathbf{1}(0, T]) \cdot M = H \cdot (M^T).
$$

Proof. There are three stages:

1. First, fix $H \in \mathcal{S}$ and $M \in \mathcal{M}_c^2$, T taking finitely many values
2. Next, fix $H \in \mathcal{S}$ and $M \in \mathcal{M}_c^2$, T a general stopping time. then approximate T using $T_{n,m} = (2^{-n}\lceil 2^n T \rceil) \wedge m$.
3. Finally, $H \in L^2(M)$, $M \in \mathcal{M}_c^2$ and T a general stopping time, and then approximate H by considering a sequence $(H^n)_{n\geq 1}$ in $\mathcal{S}$ such that $H^n \to H$ in $L^2(M)$.

□

Definition 3.10 (Locally bounded process)**.** A previsible process H is *locally bounded* if there exists a sequence $(S_n)_{n\geq 1}$ of stopping times with $S_n \uparrow \infty$ almost surely such that $H\mathbf{1}(0, S_n]$ is bounded for all n.

Definition 3.11. Let H be a locally bounded previsible process such that $H\mathbf{1}(0,S_n]$ is bounded for all n, for $(S_n)_{n\geq 1}$ a sequence of stopping times with $S_n \uparrow \infty$ a.s. and let M be a continuous local martingale iwth reduce sequence $(S'_n)_{n\geq 1}$ given by $S'_n = \inf\{t \geq 0 : |M_t| \geq n\}$ so that $M^{S'_n} \in \mathcal{M}_c^2$ for all n. Let $T_n = S_n \wedge S'_n$ for all $n \geq 1$ and define

$$(H \cdot M)_t := ((H\mathbf{1}(0,T_n]) \cdot M^{T_n})_t$$

for $t \leq T_n$.

We next show that this extension of the stochastic integral still continues to behave well under stopping.

Claim 3.12. *Let $M \in \mathcal{M}_{c,loc}$ and let H be a locally bounded previsible process. Then $H \cdot M \in \mathcal{M}_{c,loc}$ and $(T_n)_{n\geq 1}$ as given in the previous defintion is a reducing sequence. Then for any stopping time T we have*

$$(H \cdot M)^T = (H\mathbf{1}(0,T]) \cdot M = H \cdot (M^T).$$

Claim 3.13. *Let $M \in \mathcal{M}_{c,loc}$ and let H be a locally bounded previsible process. Then*

$$[H \cdot_{\text{Itô}} M] = H^2 \cdot_{\text{Lebesgue-Stieljtes}} [M].$$

Proof.

- Suppose H, M are uniformly bounded, then for all bounded stopping times T

$$\begin{aligned}\mathbb{E}[(H \cdot M)_T^2] &= \mathbb{E}[((H\mathbf{1}(0,T]) \cdot M)_\infty^2] \\ &\overset{\text{Itô's}}{=} \mathbb{E}[((H\mathbf{1}(0,T]) \cdot [M])_\infty] \\ &= \mathbb{E}[(H^2 \cdot [M])_T]\end{aligned}$$

  so by the converse of the OST it follows that $(H \cdot M)^2 - H^2 \cdot [M]$ is a continuous martingale, and by uniqueness of quadratic variation process it follows that $[H \cdot M] = H^2 \cdot [M]$.

- If H is only locally bounded then as always we employ a *localization argument*. Let (T_n) be a sequence of stopping times such that $H\mathbf{1}(0,T_n], M^{T_n}$ are uniformly bounded, then using MCT we can write

$$\begin{aligned}[H \cdot M] &= \lim_{n\to\infty} [H \cdot M]^{T_n} \\ &= \lim_{n\to\infty} [(H \cdot M)^{T_n}] \\ &= \lim_{n\to\infty} [(H\mathbf{1}(0,T_n]) \cdot M^{T_n}] \\ &= \lim_{n\to\infty} (H\mathbf{1}(0,T_n])^2 \cdot [M^{T_n}] \\ &= H^2 \cdot [M].\end{aligned}$$

□

Everything is in the same space, so we can iterate integrals.

Claim 3.14. *Let $M \in \mathcal{M}_{c,loc}$ and let H, K be localy bounded previsible processes. Then*

$$H \cdot (K \cdot M) = (HK) \cdot M.$$

Proof. These are well-defined because

$$\|H\|_{L^2(K\cdot M)} = \|HK\|_{L^2(M)}. \quad (1)$$

Let (H^n) and (K^n) be sequences of simple processes converging to H and K respectively, and then note that

$$\begin{aligned}\|H^n \cdot (K^n \cdot M) - H \cdot (K \cdot M)\| &\le \|(H^n - H) \cdot (K^n \cdot M)\| + \|H \cdot ((K^n - K) \cdot M)\| \\ &\overset{\text{Itô's}}{=} \|H^n - H\|_{L^2(K^n \cdot M)} + \|H\|_{L^2((K^n - K)\cdot M)} \\ &\overset{(1)}{=} \|(H^n - H)K^n\|_{L^2(M)} + \|H(K^n - K)\|_{L^2(M)} \\ &\le \|H^n - H\|_{L^2(M)} \|K^n\|_{L^\infty} + \|H\|_{L^\infty} \|K^n - K\|_{L^2(M)} \to 0.\end{aligned}$$

Now finish with a localization argument to handle the case where H, K are locally bounded. □

3.4 H locally bounded, M semimartingale

Definition 3.15 (Doob-Meyer decomposition of semimartingales)**.** A continuous adapted process X is a semimartingale if it can be written in the form

$$X = X_0 + M + A$$

where M is a continuous local martingale, A is a process of finite variation and $M_0 = A_0 = 0$.

Assuming that H is *left-continuous*, then we can obtain the integral as the limit of its Riemann sum approximations.

Claim 3.16. *Let X be a continuous semimartingale and H be a locally bounded left-continuous process which is a dapted. Then*

$$\sum_{k=0}^{\lfloor 2^n t \rfloor - 1} H_{k2^{-n}} (X_{(k+1)2^{-n}} - X_{k2^{-n}}) \to (H \cdot X)_t$$

ucp as $n \to \infty$.

Proof. Let $H_t^n = H_{2^{-n}\lfloor 2^n t \rfloor} = \sum_k H_{k2^{-n}} \mathbf{1}[k2^{-n}, (k+1)2^{-n})(t)$. Check that

$$(H^n \cdot M)_t = \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} H_{k2^{-n}} (M_{(k+1)2^{-n}} - M_{k2^{-n}}) + H_{2^{-n}\lfloor 2^n t \rfloor} (M_t - M_{2^{-n}\lfloor 2^n t \rfloor})$$

and then use Itô's isometry to check that $\|H^n \cdot M - H \cdot M\|^2 \to 0$. □

Remark 3.17. Itô integration effectively constructs a (local) martingale, and so we would not expect $I(\cdot)$ to preserve positivity.

4 Stochastic calculus

Definition 4.1 (Polarization identity)**.** For $M, N \in \mathcal{M}_{c,loc}$, define the *covariation* to be

$$[M, N] := \frac{1}{4}([M + N] - [M - N]).$$

Theorem 4.2. *Let $M, N \in \mathcal{M}_{c,loc}$. Then the following hold:*

(a) $[M, N]$ is the unique (up to indistinguishability) continuous, adapted, finite variation process such that $[M, N]_0 = 0$ and $MN - [M, N] \in \mathcal{M}_{c,loc}$.

(b) For $n \geq 1$, define

$$[M,N]_t^n := \sum_{k=0}^{\lceil 2^n t \rceil - 1} (M_{(k+1)2^{-n}} - M_{k2^{-n}})(N_{(k+1)2^{-n}} - N_{k2^{-n}}).$$

Then $[M,N]_t^n \to [M,N]_t$ ucp as $n \to \infty$.

(c) If $M, N \in \mathcal{M}_c^2$, then $MN - [M,N]$ is a UI martingale.

(d) For H locally bounded and previsible, it holds

$$[H \cdot M, N] + [M, H \cdot N] = 2H \cdot [M,N].$$

Proof. For (d), expand $(H+1)^2 \cdot [M,N] = [(H+1) \cdot M, (H+1) \cdot N]$. □

Theorem 4.3 (Kunita-Watanabe)**.** *Let $M, N \in \mathcal{M}_{c,loc}$ and H a locally bounded, previsible process. Then*

$$[H \cdot M, N] = H \cdot [M,N].$$

Proof. It suffices to show $[H \cdot M, N] = [M, H \cdot N]$ and then invoke (d) from earlier. By uniqueness of quadratic variation, it suffices to prove that $(H \cdot M)N - M(H \cdot N) \in \mathcal{M}_{c,loc}$. By localization it suffices to restrict to $M, N \in \mathcal{M}_c^2$ and H bounded, and by OST it suffices to prove $\mathbb{E}[(H \cdot M)_T N_T] = \mathbb{E}[M_T (H \cdot N)_T]$ for all bounded stopping times T. But by some shimmy-ing once more it suffices to prove $\mathbb{E}[(H \cdot M)_\infty N_\infty] = \mathbb{E}[M_\infty (H \cdot N)_\infty]$ for all $M, N \in \mathcal{M}_c^2$ and H bounded.

When $H = Z\mathbf{1}(s,t]$ where Z is bounded and $\mathcal{F}_s$-measurable we can check that the above evaluates to $\mathbb{E}[Z(M_t N_t - M_s N_s)]$. Use linearity to extend to $H \in \mathcal{S}$. Now approximate H by $H^n \to H$ in $L^2(M)$ and $L^2(N)$ then we can show that $(H^n \cdot M)_\infty N_\infty$ converges in expectation to $(H \cdot M)_\infty N_\infty$ by Cauchy-Schwarz or whatever. □

Definition 4.4. For X, Y continuous semimartingales we define their covariation $[X,Y]$ to just be the covariation of their respective martingale parts in the Doob-Meyer decomposition.

Claim 4.5. *Let X, Y be independent continuous semimartingales. Then $[X,Y] = 0$.*

4.1 Itô's formula

Claim 4.6. *Let X and Y be continuous semimartingales. Then*

$$X_t Y_t - X_0 Y_0 = \int_0^t X_s dY_s + \int_0^t Y_s dX_s + [X,Y]_t.$$

Proof. It basically following by dyadically summing the following identity: For $s \leq t$, we have

$$X_t Y_t - X_s Y_s = X_s(Y_t - Y_s) + Y_s(X_t - X_s) + (X_t - X_s, Y_t - Y_s).$$

□

Theorem 4.7 (Itô's formula)**.** *Let $X^1, \ldots, X^d$ be continuous semimartingales and let $X = (X^1, \ldots, X^d)$. Let $f \colon \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{C}^2$. Then*

$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(X_s) dX_s + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j} d[X^i, X^j]_s.$$

Remark 4.8. A consequence of Itô's is that if f is harmonic then $f(B_t) \in \mathcal{M}_{c,loc}$; further if f is bounded then $f(B_t)$ is a (true) martingale.

Define the *Stratonovich integral* as

$$\int_0^t X_s \partial Y_s = \int_0^t X_s dY_s + \frac{1}{2}[X,Y]_t.$$

The Riemann sum approximations for the term on the RHS is midpoint rather than the left endpoint

$$\sum_{k=0}^{\lfloor 2^n t\rfloor -1} \left(\frac{X_{(k+1)2^{-n}} + X_{k2^{-n}}}{2} \right) (Y_{(k+1)2^{-n}} - Y_{k2^{-n}}).$$

However, because the integrand is the midpoint, we end up with the fact that the Stratonovich integral is no longer necessarily a local martingale. The perk is that the integration by parts formula is particularly simple:

$$X_t Y_t - X_0 Y_0 = \int_0^t X_s \partial Y_s + \int_0^t Y_s \partial X_s.$$

The actual thing to remember is the following list of shorthand:

- (iterated integral) $H_t d(K_t dX_t) = (H_t K_t) dX_t$
- (Kunita-Watanabe) $H_t dX_t dY_t = d(H_t dX_t) dY_t$
- (IBP) $d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t$
- (Itô's) $df(X_t) = \sum_{i=1}^d \frac{\partial f}{\partial x_i}(X_t) dX_t^i + \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j}(X_t) dX_t^i dX_t^j$.

5 Applications

Theorem 5.1 (Lévy's characterization of BM)**.** *Let $X^1, \ldots, X^d$ be continuous local martingales and set $X = (X^1, \ldots, X^d)$. Suppose that $X_0 = 0$ and that $[X^i, X^j]_t = \delta_{ij} t$ for all i, j and $t \geq 0$. Then X is a standard Brownian motion on $\mathbb{R}^d$.*

Proof. It suffices to show that for all $\theta \in \mathbb{R}^d$,

$$\mathbb{E}[e^{i\langle \theta, X_t - X_s\rangle} \mid \mathcal{F}_s] = \exp\left(-\frac{1}{2}|\theta|^2 (t-s) \right).$$

This is basically equivalent to proving that $Z_t = \exp(i\langle \theta, X_t\rangle + \frac{1}{2}|\theta|^2 t)$. We will show that $Z \in \mathcal{M}_{c,loc}$ and this suffices since Z is bounded on $[0,t]$ for each $t \geq 0$.

By definition, $Y_t = \langle \theta, X_t \rangle$ has the property that

$$[Y]_t = |\theta|^2 t.$$

Now, $Z_t = \exp(iY_t + \frac{1}{2}[Y]_t)$ and then we apply Itô's with $f(x,y) = \exp(ix + y/2)$ to get that $dZ_t = iZ_t dY_t - \frac{1}{2} Z_t \cdot [Y_t] + \frac{1}{2} Z_t \cdot [Y_t] = iZ_t dY_t$ and then it follows that $Z_t \in \mathcal{M}_c$, as desired. ☐

A consequence is that all continuous local martingales and martingales are time-changed Brownian motions.

Theorem 5.2 (Dubins-Schwarz)**.** *Let $M \in \mathcal{M}_{c,loc}$ with $M_0 = 0$ and $[M]_\infty = \infty$. Set $\tau_s := \inf\{t \geq 0 : [M]_t > s\}$, $B_s := M_{\tau_s}$ and $\mathcal{G}_s := \mathcal{F}_{\tau_s}$. Then τ_s is an $(\mathcal{F}_t)$-stopping time and $[M]_{\tau_s} = s$ for all $s \geq 0$. Moreover, B is a $(\mathcal{G}_s)$-Brownian motion and*

$$M_T = B_{[M]_t}.$$

One immediate point of contention is that $[M]_t$ is potentially flat on some intervals. This seems to suggest that there is no hope for Dubins-Schwarz to work because $B_{[M]_t}$ is plausibly discontinuous. This issue is handled by the flatness lemma which states that M and $[M]$ are constant on the same intervals.

Lemma 5.3 (Flatness lemma)**.** *A.s. for all* $0 \leq a < b$, *for all* $t \in [a, b]$, *we have that* $M_t = M_a$ *iff* $[M]_b = [M]_a$.

Proof. Basically follows from martingale property of $M^2 - [M]$ and M itself. Let $S_q = \inf\{t > q : [M]_t > [M]_q\}$ and so M is constant on $[q, S_q]$. OST effectively says that $\mathbb{E}[M_{S_q}^2 - M_q^2 \mid \mathcal{F}_q] = [M]_{S_q} - [M]_q$. Consequently, using orthogonality of martingale increments of M it follows that

$$\mathbb{E}[(M_{S_q} - M_q)^2 \mid \mathcal{F}_q] = \mathbb{E}[M_{S_q}^2 - M_q^2 \mid \mathcal{F}_q] = \mathbb{E}[[M]_{S_q} - [M]_q \mid \mathcal{F}_q] = 0.$$

□

Proof of Theorem 5.2.

- (τ_s is a stopping time, ...) $[M]$ is continuous and adapted. $[M]_\infty = \infty$ implies that $\tau_S < \infty$ for all $s \geq 0$. Apply other known abstract nonsense from Advanced Probability if necessary.
- (B is continuous) $s \mapsto \tau_s$ is non-decreasing and cadlag. It suffices to show that $B_{s^-} = B_s$ for all $s \geq 0$. If $\tau_s = \tau_{s^-}$ we are done. Else, by localization $M \in \mathcal{M}_c^2$ and then apply Lemma 5.3 to the interval $[\tau_{s^-}, \tau_s]$ on which $[M]$ is flat.
- (B is BM) Since $[M^{\tau_s}]_\infty = [M]_{\tau_s} = s$, and consequently $M^{\tau_s} \in \mathcal{M}_c^2$ since $\mathbb{E}[[M^{\tau_s}]_\infty] < \infty$. In particular, we have that $(M^2 - [M])^{\tau_s}$ is a UI martingale. This immediately shows that the OST that B is a martingale with $[B]_t = t$ and then we can just apply the Lévy characterization.

□

Example 5.4 (Extension of an earlier remark)**.** Let $h \in L^2([0, \infty))$ and let $M_t := \int_0^t h(s) dB_s$. Then $M_0 = 0$, $M \in \mathcal{M}_{c,loc}$ and $[M]_t = \int_0^t h^2(s) ds$ and then

$$M_\infty \overset{(d)}{=} B_{\int_0^\infty h^2(s) ds} \sim N(0, \|h\|_{L^2}^2).$$

Corollary 5.5 (Dudley?)**.** *For any* $0 \leq a < b$ *and any finite random variable* $X \in \mathcal{F}_a$, *there is a finite stopping time* τ *with* $a \leq \tau < b$ *such that*

$$X = \int_a^\tau \frac{1}{b - t} dB_t.$$

Definition 5.6 (Exponential martingale)**.** Let $M \in \mathcal{M}_{c,loc}$ with $M_0 = 0$. Define the process $\mathcal{E}(M)_t$ by setting

$$\mathcal{E}(M)_t = \exp(M_t - \tfrac{1}{2}[M]_t).$$

Then $\mathcal{E}(M)$ is the *stochastic exponential* of M. Note that $\mathcal{E}(M) \in \mathcal{M}_{c,loc}$ and it satisfies $d\mathcal{E}(M)_t = \mathcal{E}(M)_t dM_t$.

Here $\mathcal{E}(M) \in \mathcal{M}_{c,loc}$ because by Itô's, we can write $dZ_t = Z_t(dM_t - \frac{1}{2} d[M]_t) + \frac{1}{2} Z_t d[M]_t = Z_t dM_t$.

Theorem 5.7. *Let* $M \in \mathcal{M}_{c,loc}$ *with* $M_0 = 0$. *Suppose that* $[M]$ *is uniformly bounded, then* $\mathcal{E}(M)$ *is a UI martingale.*

If the quadratic variation of a continuous local martingale is small then the martingale cannot be too large.

Claim 5.8. *Let $M \in \mathcal{M}_{c,loc}$ with $M_0 = 0$. Then for all $\varepsilon, \delta > 0$, we have that*

$$\mathbb{P}\left[\sup_{t\geq 0} M_t \geq \varepsilon, [M]_\infty \leq \delta\right] \leq \exp\left(-\frac{\varepsilon^2}{2\delta}\right).$$

Proof. Fix $\varepsilon > 0$ and let $T = \inf\{t \geq 0 : M_t \geq \varepsilon\}$. Fix $\theta > 0$ and set $Z_t = \exp\left(\theta M_t^T - \frac{1}{2}\theta^2 [M]_t^T\right)$ then $Z_t \in \mathcal{M}_{c,loc}$ since $Z = \mathcal{E}(\theta M^T)$ and $\theta M^T \in \mathcal{M}_{c,loc}$. Moreover, $|Z_t| \leq e^{\theta\varepsilon}$ for all $t \geq 0$ by construction so Z is a bounded martingale. And consequently $\mathbb{E}[Z_\infty] = Z_0 = 1$. For each $\delta > 0$, we have

$$\begin{aligned}
\mathbb{P}\left[\sup_{t\geq 0} M_t \geq \varepsilon, [M]_\infty \leq \delta\right] &= \mathbb{P}\left[\sup_{t\geq 0} e^{\theta M_t^T} \geq e^{\theta\varepsilon}, [M]_\infty \leq \delta\right] \\
&\leq \mathbb{P}\left[\sup_{t\geq 0} Z_t \geq e^{\theta\varepsilon - \theta^2\delta/2}\right] \\
&\leq \exp(-\theta\varepsilon + \theta^2\delta/2).
\end{aligned}$$

where the last step is by Doob's maximal inequality. □

Proof of Theorem 5.7. It suffices to show that $\mathcal{E}(M)$ is uniformly bounded by an integrable random variable; we use the bound $\sup_{t\geq 0} \mathcal{E}(M)_t \leq \exp\left(\sup_{t\geq 0} M_t\right)$, since $[M]_t \geq 0$ for all $t \geq 0$. Now use the previous lemma to write

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\sup_{t\geq 0} M_t\right)\right] &= \int_0^\infty \mathbb{P}\left[\exp\left(\sup_{t\geq 0} M_t\right) \geq \lambda\right] d\lambda \\
&= \int_0^\infty \mathbb{P}\left[\sup_{t\geq 0} M_t \geq \log\lambda\right] d\lambda \\
&\leq 1 + \int_1^\infty \exp\left(-\frac{(\log\lambda)^2}{2c}\right) d\lambda < \infty.
\end{aligned}$$

□

Next, we see that a change in measure is synonymous to a change in drift.

Theorem 5.9 (Girsanov's)**.** *Let $M \in \mathcal{M}_{c,loc}$ be such that $M_0 = 0$. Suppose that $Z = \mathcal{E}(M)$ is a UI martingale. Then we can define a new probability measure $\widetilde{\mathcal{P}} \ll \mathcal{P}$ on $(\Omega, \mathcal{F})$ by setting $\widetilde{\mathcal{P}} = \mathbb{E}[Z_\infty \mathbf{1}(A)]$ fro $a \in \mathcal{F}$. If $X \in \mathcal{M}_{c,loc}(\mathbb{P})$ then $X - [X, M] \in \mathcal{M}_{c,loc}(\widetilde{\mathbb{P}})$.*

Proof. Let $T_n = \inf\{t \geq 0 : |X_t - [X, M]_t| \geq n\}$. It suffices to show that $Y^{T_n} := X^{T_n} - [X^{T_n}, M] \in \mathcal{M}_c(\widetilde{\mathbb{P}})$ for all n. Now, the key property of the exponential martingale is that Itô's gives $dZ_t = Z_t dM_t$, and so by IBP we get

$$\begin{aligned}
d(Z_t Y_t) &= Y_t dZ_t + Z_t dY_t + dY_t dZ_t \\
&= Y_t dZ_t + Z_t dX_t
\end{aligned}$$

which implies that $Z_T Y_T \in \mathcal{M}_{c,loc}(\mathbb{P})$. To finish up it suffices to note that Z_T is UI while Y is bounded. □

Remark 5.10. The quadratic variation does not change under this change of measures from $\mathbb{P}$ to $\widetilde{\mathbb{P}}$.

As a consequence of the above remark and Lévy's characterization of BM, it follows that we have the following.

Corollary 5.11. *Let B be a standard Brownian motion under $\mathbb{P}$ and let $M \in \mathcal{M}_{c,loc}$ with $M_0 = 0$. Suppose that $Z = \mathcal{E}(M)$ is a UI martingale and $\widetilde{\mathbb{P}}(A) = \mathbb{E}[Z_\infty \mathbf{1}(A)]$ for $A \in \mathcal{F}$. Then $\widetilde{B} = B - [B, M]$ is a $\widetilde{\mathbb{P}}$-Brownian motion.*

In more concrete terms, we have the following variant of Girsanov's.

Corollary 5.12 (Restatement of Girsanov's in the Brownian setting)**.** *If B_t is a Brownian motion under a measure P then by considering the measure Q with Radon-Nikodym derivative*

$$\frac{dQ}{dP} = \exp\left(\int_0^t \mu_s dB_s - \frac{1}{2}\int_0^t \mu_s^2 ds\right),$$

and under this new measure Q,

$$\widetilde{B_t} = B_t - \int_0^t \mu_s ds.$$

is a Brownian motion.

Remark 5.13. As a concrete example, consider the following SDE with drift

$$dX_t = \mu dt + \sigma dB_t.$$

Then the change of measure given by

$$\frac{dQ}{dP} = \exp\left(-\frac{\mu}{\sigma}B_t - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 t\right)$$

removes the drift.

6 Stochastic differential equations

Let $\sigma\colon \mathbb{R}^d \to \mathscr{M}^{d\times m}(\mathbb{R})$ and $b\colon \mathbb{R}^d \to \mathbb{R}^d$ measurable, and consider

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt.$$

A solution is given by:

- A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ where $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions.
- An $(\mathcal{F}_t)$-Brownian motion in $\mathbb{R}^m$,
- An $(\mathcal{F}_t)$-adapted continuous process X in $\mathbb{R}^d$ such that

$$X_t = X_0 + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds.$$

Definition 6.1. Let $(B_t)_{t\geq 0}$ be a Brownian motion with admissible filtration $(\mathcal{F}_t)_{t\geq 0}$. Then we say that $(X_t, \mathcal{F}_t)$ is a *strong solution with initial condition* x_0 if

$$X_T - X_0 = \int_0^t \sigma(s, X_s)dB_s + \int_0^t b(s, X_s)ds$$

and $X_0 = x_0$ holds a.s. for all $t \geq 0$.

For strong solutions, we can define *pathwise uniqueness* which occurs if X and X' are solutions then $\mathbb{P}[X_t = X_t' \ \forall t \geq 0] = 1$ and fundamentally this is because we fixed the underlying probability space.

Definition 6.2. $(X_T, \mathcal{F}_t)$ on some probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ is a *weak solution with initial distribution* μ if there exists a Brownian motion $(B_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $(\mathcal{F}_t)_{t\geq 0}$ is an admissible filtration, $\mathbb{P}(X_0 \in \cdot) = \mu(\cdot)$ and

$$X_t - X_0 = \int_0^t \sigma(s, X_s) dB_s + \int_0^t b(s, X_s) ds$$

holds a.s. for all $t \geq 0$.

For weak solutions, we ask for *uniqueness in law* which is when all solutions to the SDE starting from x_0 have the same distribution.

Example 6.3. The SDE $dX_t = -\text{sgn}(X_t) dB_t$, $X_0 = 0$ has a weak solution with unique law but no strong solution with pathwise uniqueness.

Theorem 6.4. *If σ and b are* Lipschitz *then there is pathwise uniqueness; for each probability space satsifying the usual condition and each choice of $\mathcal{F}_t$-Brownian motion B, there is a strong solution from any $x \in \mathbb{R}^d$.*

Here for σ we measure Lipschitzness in terms of the Frobenius norm.

Existence follows from contraction mapping, and uniquess follows from Gronwall's:

Lemma 6.5 (Gronwall's)**.** *Let $T > 0$ and let f be a non-negative, bounded, measurable function on $[0, T]$. Suppose that there exists $a, b \geq 0$ such that for all $t \in [0, T]$ we have*

$$f(t) \leq a + b \int_0^t f(s) ds,$$

Then $f(t) \leq ae^{bt}$.

The way to remember this lemma is that if $f(t)$ satisfies a (differential or integral) inequality of a suitable type, then this limits the growth of $f(t)$ in such a way that $f(t)$ can become at most as big as the function $f(t)$ which satisfies the corresponding equality. To solve for equality we use the ODE trick of an *integrating factor*.

Proof.

1. (Pathwise uniqueness) Let $\tau = \inf\{t \geq 0 : |X_t| \vee |X'_t| \geq M\}$ and then we can use Itô's and C-S to bound

$$\begin{aligned}
\mathbb{E}[(X_{t\wedge\tau} - X'_{t\wedge\tau})^2] &\leq 2\mathbb{E}\left[\left(\int_0^{t\wedge\tau} (\sigma(X_s) - \sigma(X'_s)) dB_s\right)^2\right] + 2\mathbb{E}\left[\left(\int_0^{t\wedge\tau} b(X_s) - b(X'_s)\right)^2\right] \\
&\leq 2\mathbb{E}\left[\int_0^{t\wedge\tau} (\sigma(X_s) - \sigma(X'_s))^2 ds\right] + 2T\mathbb{E}\left[\int_0^{t\wedge\tau} (b(X_s) - b(X'_s))^2\right] \\
&\leq 2K^2(1+T) \int_0^t \mathbb{E}[(X_{s\wedge\tau} - X_{s\wedge\tau};)^2] ds.
\end{aligned}$$

   Then apply Gronwall's to $f(t) = \mathbb{E}[(X'_{t\wedge\tau)-X_{t\wedge\tau}})^2]$.

2. (Existence of strong solution) We show that a solution can be given as the fixed point of a contraction mapping

$$F(X)_t = x + \int_0^t \sigma(X_s) dB_S + \int_0^t b(X_s) ds.$$

   We can apply Doob's L^2 inequality to bound (M_t) where $M_t = \int_0^t \sigma(X_s) dB_s$ because $\mathbb{E}([M]_T) \leq 2T(|\sigma(0)|^2 + K^2 \|\!|X|\!\|_T^2) < \infty$. Anyways, combining Doob's and C-S easily show that

$$\left|\!\left|\!\left| F^{(n)}(X) - F^{(n)}(Y) \right|\!\right|\!\right|_T^2 \leq \frac{C_T^n T^n}{n!} \|\!|X - Y|\!\|_T^2.$$

To show that this is $\mathcal{F}_t^B$ adapted, it suffices to take $Y^0 = x$ and let $Y^n = F(Y^{n-1})$. Here Y^n is evidently $\mathcal{F}_t^B$ adapted, and by the earlier inequality we have that $|||X - Y|||_T^2 \le \frac{C_T^n T^n}{n!} |||X - x|||_T^2$, so $Y^n \to X$ in $\mathcal{C}_T$ and so there exists a subsequence $(Y^{n_k})_{k\ge1}$ such that $Y^{n_k} \to X$ uniformly on $[0,T]$. Since X is the a.s. limit of $\mathcal{F}_t^B$ random variables, it follows that it must be as well.

3. (Uniqueness in law) As we iterate, it is enough to note that $Y^n \to X$ and $\widetilde{Y}^n \to \widetilde{X}$ uniformly on compact time intervals, and then we can use induction to prove that $Y^n \sim \widetilde{Y}^n$.

□

Definition 6.6. A *locally defined process* $(X, \mathcal{T})$ where $\mathcal{T}$ is a stopping time and $X\colon \{(\omega,t) \in \Omega \times [0,\infty) : t < \mathcal{T}(\omega)\} \to \mathbb{R}$.

Definition 6.7. We say that $(X,\mathcal{T})$ is a *maximal* local solution to an SDE if for any other local solution (Y,η) on the same space such that $X_t = Y_t$ for all $t < T \wedge \eta$, we have that $\eta \le T$.

Definition 6.8. Let $U \subset \mathbb{R}^d$ be open. Then $f\colon U \to \mathbb{R}^d$ is *locally Lipschitz* if for each compact set $C \subset U$, we have that $f\big|_C$ is Lipschitz.

Theorem 6.9. *Let $U \subset \mathbb{R}^d$ be open and suppose $\sigma\colon U \to \mathscr{M}^{d\times m}(\mathbb{R})$ and $b\colon U \to \mathbb{R}^d$ are locally Lipschitz. Then for all $x \in U$, the SDE $dX_t = \sigma(X_t)dB_t + b(X_t)dt$ has a pathwise unique maximal local solution $(X,\mathcal{T})$ starting from x. Moreover, for all compact sets $C \subset U$, on the event $\{\mathcal{T} < \infty\}$, we have that*

$$\sup\{t < \mathcal{T} : X_t \in C\} < T.$$

Proof. The strategy of the proof is as follows:

1. Fix $C \subset U$ and let $T = \inf\{t \ge 0\colon \widetilde{X}_t \in C\}$. There exists a globally Lipschitz function that agrees with σ, b on C. Use our earlier work to show that there is a unique local solution in C.

2. To build a local solution on the whole space, approximate U by growing compact sets C_n and "glue" the corresponding local solutions.

3. For maximality, suppose we have two maximal solutions $(X,\mathcal{T})$ and (Y,η). Then define $S_n = \inf\{t \le \eta : Y_t \notin C_n\} \wedge \eta$. Now earlier work shows that $X_t = Y_t$ for all $t < T_n \wedge S_n$, so $S_n \le T_n$. Let $n \to \infty$ to show that $t \le \eta$.

4. Finally, we need to show that we get well-defined solutions on each compact. To this end one can show that if we nest C in another compact C', that the number of crossing that X makes from C_2 to C_1 can be written as the solution to a SDE with uniformly bounded coefficients, so it is easily to conclude by Borel-Cantelli that crossings from C to C' cannot occur infinitely often.

□

7 Diffusion processes

It turns out that solutions to SDEs is very much related to martingale problems. Given bounded, measurable $a\colon \mathbb{R}^d \to \mathscr{M}^{d\times d}(\mathbb{R})$ and $b\colon \mathbb{R}^d \to \mathbb{R}^d$ with a symmetric and $f \in C_b^2(\mathbb{R})$, write

$$Lf(x) := \sum_{i=1}^d b_i(x)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x).$$

Definition 7.1. X is an *L-diffusion* if for all $f \in C_b^2(\mathbb{R}^d)$, we have

$$M_t^f := f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds.$$

Example 7.2. A restatement of Itô's formula is that if $a = \sigma\sigma^T$ then $X_t = \sigma B_t + bt$ is an (a, b)-diffusion.

L-diffusions and SDEs are intimately related. One direction follows from Itô's almost immediately, and for the other direction one combines the result of being an L-diffusion for $f \equiv x^2$ together with Itô's applied to X_t in order to show that $N_t := X_t - X_0 - \int_0^t b(X_t)ds$ has the property that $[N]_t = \int_0^s \sigma^2(X_s)ds$ and then we can by hand construct a Brownian motion B_s (check using Lévy's characterization of Brownian motion!) that $N = \int_0^s \sigma(X_s)dB_s$.

Theorem 7.3. *Suppose that X is a solution to the SDE $dX_t = \sigma(X_t)dB_t + b(X_t)dt$. Let $f \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$. Then the process*

$$M_t^f = f(t, X_t) - f(0, X_0) - \int_0^t \left(\frac{\partial}{\partial s} + L\right) f(s, X_s)ds$$

is a continuous local martingale where $a = \sigma\sigma^T$ and L is as defined above. In particular, if σ, b are bounded then X is an L-diffusion.

It turns out that restarting an L-diffusion at a finite stopping time again gives an L-diffusion.

Theorem 7.4. *Let X be an L-diffusion and T a fintie stopping time. Set $\widetilde{X_t} = X_{T+t}$ and $\widetilde{\mathcal{F}}_t = \mathcal{F}_{T+t}$. Then $\widetilde{X}$ is an L-diffusion with respect to $\widetilde{\mathcal{F}}_t$.*

Proof. Use OST to show that $\mathbb{E}[(\widetilde{M}_t^f - \widetilde{M}_s^f)\mathbf{1}(A \cap \{T \leq n\})] = 0$ and then let $n \to \infty$ and use DCT to conclude. ☐

What's the relation between L-diffusions and martingale problems?

Lemma 7.5. *Let X be an L-diffusion. Then for all $f \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$, the process*

$$M_t^f = f(t, X_t) - f(0, X_0) - \int_0^t \left(\frac{\partial}{\partial s} + L\right) f(s, X_s)ds$$

is a martingale.

Definition 7.6. We say that a is *uniformly positive definite* (UPD) if there exists $\varepsilon > 0$ such that for all $x, \xi \in \mathbb{R}^d$, we have

$$(\xi, a(x)\xi) \geq \varepsilon^2 |\xi|^2.$$

Effectively our earlier results allows us to use SDEs to build (a, b)-diffusions, and then we can use the lemma above to also get corresponding martingales. As a generalization to the idea of what we have seen in Lattice Models/Advanced Probability of using Brownian motion to solve boundary value problems to $\Delta = 0$, we have the following two theorems.

Theorem 7.7 (Dirichlet problem)**.** *Suppose that $u \in C(\overline{D}) \cap C^2(D)$ satisfies*

$$\begin{cases} Lu + \varphi = 0 & \text{on } D, \\ u = f & \text{on } \partial D, \end{cases}$$

with $f \in C(\partial D)$, $\varphi \in C(\overline{D})$. Then for any L-diffusion X starting from $x \in D$, we have that

$$u(x) = \mathbb{E}_x \left[\int_0^T \varphi(X_s)ds + f(X_T) \right]$$

where $T = \inf\{t \geq 0 : X_t \notin D\}$*. In particular, for all Borel sets* $A \subset D$*, and* $B \subset \partial D$*, we have that*

$$\begin{cases} \mathbb{E}_x\left[\int_0^T \mathbf{1}(X_s \in A) ds\right] = \int_A g(x,y)dy \\ \mathbb{P}_x(X_T \in B) = \int_B m(x,y)\lambda(dy) \end{cases}$$

Theorem 7.8 (Cauchy problem)**.** *Assume that* $f \in C_b^2(\mathbb{R}^d)$*. Let* $u \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ *satisfy*

$$\begin{cases} \frac{\partial u}{\partial t} = Lu & \text{on } \mathbb{R}_+ \times \mathbb{R}^d, \\ u(0,\cdot) = f \, \text{on } \mathbb{R}^d. \end{cases}$$

Let $p\colon (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to (\infty)$ *be the corresponding heat Then for any* L*-diffusion* X*, for all* $t \in \mathbb{R}_+$*,* $x \in \mathbb{R}^d$*,* $s \leq t$*, we have that* $\mathbb{E}[f(X_t) \,|\, \mathcal{F}_s] = u(t-s, X_s)$ *a.s. In particular*

$$\mathbb{E}_x[f(X_t)] = u(t,x) = \int_{\mathbb{R}^d} p(t,x,y) f(y) dy.$$

In fact, X *is a Markov process with transition density function* p*.*